International Journal of Theoretical Physics, Vol. 31, No. 8, 1992

Bianchi Type III and Kantowski–Sachs Cosmological Models in Lyra Geometry

T. Singh[1] and G. P. Singh[1]

Received August 9, 1990

Cosmological models for Bianchi type III and Kantowski-Sachs space-times within the framework of Lyra geometry are obtained. The physical behavior of the models is also discussed.

1. INTRODUCTION

Lyra (1951) suggested a modification of Riemannian geometry by introducing a gauge function which is a metrical concept in Weyl (1918) geometry in the geometrical structureless manifold. Subsequent investigations were done by Sen (1957, 1960), Halford (1970), Sen and Dunn (1971), Bhamra (1974), Beesham (1986), and Soleng (1987) in scalar-tensor theory and cosmology within the framework of Lyra geometry. Singh and Singh (1991*a,b*) considered Bianchi type I, V, and VI_0 cosmological models in the Lyra geometry.

Several authors (Stewart and Ellis, 1968; Cohen and Defrise, 1968; Vajk and Eltgroth, 1970; Moussiaux *et al.*, 1981; Lorenz, 1982, 1983) have studied Bianchi type III cosmological models. Weber (1984, 1985) has done a qualitative study of Kantowski-Sachs cosmological models. Lorenz (1983), Grøn (1986), and Matravers (1988) have also studied cosmological models for Kantowski-Sachs space-time.

In this paper Bianchi type III and Kantowski-Sachs cosmological models are investigated in Lyra geometry.

2. FIELD EQUATIONS

The field equations in normal gauge for Lyra's manifold as obtained by Sen (1957) are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\tfrac{3}{2}\phi_\mu\phi_\nu-\tfrac{3}{4}g_{\mu\nu}\phi_\alpha\phi^\alpha=-\chi T_{\mu\nu} \tag{2.1}$$

[1]Department of Applied Mathematics, Institute of Technology, Banaras Hindu University, Varanasi 221005, India.

where ϕ_μ is a displacement field defined as

$$\phi_\mu = (0, 0, 0, \beta) \tag{2.2}$$

and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu - pg_{\mu\nu} \tag{2.3}$$

Perfect fluid matter is characterized by the equation of state

$$p = (\lambda - 1)\rho, \qquad 1 \leq \lambda \leq 2 \tag{2.4}$$

We are considering only nonprivileged models, i.e., an observer has comoving velocity $u^\mu = \delta^\mu_4$. We take the metric in the form

$$ds^2 = dt^2 - R_1^2\, dx^2 - R_2^2[d\theta^2 + f^2(\theta)\, d\phi^2] \tag{2.5}$$

where

$$R = R_1(t) \qquad \text{and} \qquad R_2 = R_2(t)$$

The field equation (2.1) reduces to

$$\frac{2\ddot{R}_2}{R_2} + \left(\frac{\dot{R}_2}{R_2}\right)^2 - \frac{1}{R_2^2 f}\frac{d^2 f}{d\theta^2} = -\chi p - \frac{3}{4}\beta^2 \tag{2.6}$$

$$\frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\dot{R}_1\dot{R}_2}{R_1R_2} = -\chi p - \frac{3}{4}\beta^2 \tag{2.7}$$

$$\left(\frac{\dot{R}_2}{R_2}\right)^2 + \frac{2\dot{R}_1\dot{R}_2}{R_1R_2} - \frac{1}{fR_2^2}\frac{d^2 f}{d\theta^2} = \chi\rho + \frac{3}{4}\beta^2 \tag{2.8}$$

The energy conservation equation is

$$\chi\dot{\rho} + \frac{3}{2}\beta\dot{\beta} + \left[\chi(p+\rho) + \frac{3}{2}\beta^2\right]\left(\frac{\dot{R}_1}{R_1} + \frac{2\dot{R}_2}{R_2}\right) = 0 \tag{2.9}$$

Here the quantities with dots refer to their derivatives with respect to coordinate *t*.

3. BIANCHI TYPE III SPACE-TIME

3.1. $\beta = \beta(t)$

The Bianchi type III metric is defined by

$$ds^2 = dt^2 - R_1^2\, dx^2 - R_2^2(d\theta^2 + \sinh^2\theta\, d\phi^2) \tag{3.1}$$

which can be obtained from (2.5) when $f(\theta)=\sinh\theta$. The set of field equations (2.6)-(2.9) reduce to

$$\frac{2\ddot{R}_2}{R_2}+\left(\frac{\dot{R}_2}{R_2}\right)^2-\frac{1}{R_2^2}=-\chi p-\frac{3}{4}\beta^2 \tag{3.2}$$

$$\frac{\ddot{R}_1}{R_1}+\frac{\ddot{R}_2}{R_2}+\frac{\dot{R}_1\dot{R}_2}{R_1R_2}=-\chi p-\frac{3}{4}\beta^2 \tag{3.3}$$

$$\left(\frac{\dot{R}_2}{R_2}\right)^2+\frac{2\dot{R}_1\dot{R}_2}{R_1R_2}-\frac{1}{R_2^2}=\chi\rho+\frac{3}{4}\beta^2 \tag{3.4}$$

$$\chi\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+\left[\chi(p+\rho)+\frac{3}{2}\beta^2\right]\left(\frac{\dot{R}_1}{R_1}+\frac{2\dot{R}_2}{R_2}\right)=0 \tag{3.5}$$

By a combination of equations (3.2)-(3.4), we get

$$\frac{\ddot{R}_1}{R_1}+\frac{2\dot{R}_1\dot{R}_2}{R_1R_2}=\frac{\chi}{2}(\rho-p) \tag{3.6}$$

Adding equations (3.3) and (3.4), we obtain

$$\frac{\ddot{R}_1}{R_1}+\frac{\ddot{R}_2}{R_2}+\left(\frac{\dot{R}_2}{R_2}\right)^2+\frac{3\dot{R}_1\dot{R}_2}{R_1R_2}-\frac{1}{R_2^2}=\chi(\rho-p) \tag{3.7}$$

It is difficult to solve these equations in the present form. We therefore introduce new variables η and h as

$$dt=R_2\,d\eta \tag{3.8}$$

$$h=R_1R_2 \tag{3.9}$$

Considering the equation of state (2.4) and equations (3.8) and (3.9), we can write equations (3.4)-(3.7) as

$$\left(\frac{R_2'}{R_2}\right)^2+\frac{2R_1'R_2'}{R_1R_2}-1=\left(\chi\rho+\frac{3}{4}\beta^2\right)R_2^2 \tag{3.10}$$

$$\chi\rho'+\frac{3}{2}\beta\beta'+\left(\chi\lambda\rho+\frac{3}{2}\beta^2\right)\left(\frac{h'}{h}+\frac{R_2'}{R_2}\right)=0 \tag{3.11}$$

$$\frac{R_1''}{R_1}-\left(\frac{R_1'}{R_1}\right)^2+\frac{h'R_1'}{hR_1}=\frac{\chi}{2}(2-\lambda)\rho R_2^2 \tag{3.12}$$

$$\frac{h''}{h}-1=\chi(2-\lambda)\rho R_2^2 \tag{3.13}$$

Here primes denote differentiation with respect to η. Further, it is difficult to find a general solution of the set of field equations (3.10)-(3.13). Hence we consider some particular cases of physical interest.

Case I. Empty Universe

First we consider matter-free space-time ($p = \rho = 0$). Then equations (3.10)-(3.13) become

$$\left(\frac{R_2'}{R_2}\right)^2 + \frac{2R_1'R_2'}{R_1R_2} - 1 = \frac{3}{4}\beta^2 R_2^2 \tag{3.14}$$

$$\frac{\beta'}{\beta} + \left(\frac{h'}{h} + \frac{R_2'}{R_2}\right) = 0 \tag{3.15}$$

$$\frac{R_1''}{R_1} - \left(\frac{R_1'}{R_1}\right)^2 + \frac{h'R_1'}{hR_1} = 0 \tag{3.16}$$

$$h'' - h = 0 \tag{3.17}$$

Equation (3.17) gives the general solution

$$h = m_1 \sinh(\eta + m_2) \tag{3.18}$$

With the help of equation (3.18), we obtain from equation (3.16)

$$R_1 = m_4\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{m_3} \tag{3.19}$$

Substituting the values of h and R_1 from equations (3.18) and (3.19) in equation (3.9), we get

$$R_2 = \frac{m_1}{m_4}\sinh(\eta + m_2)\left[\coth\left(\frac{\eta + m_2}{2}\right)\right]^{m_3} \tag{3.20}$$

Using equations (3.18) and (3.20) in equation (3.15) and then integrating, we get

$$\beta = m_5[\operatorname{cosech}(\eta + m_2)]^2\left[\tanh\left(\frac{\eta + m_2}{2}\right)\right]^{m_3} \tag{3.21}$$

Here m_i, $i = 1, 2, \ldots, 5$, are arbitrary constants related by

$$3m_1^2 m_5^2 + 4(m_3^2 - 1)m_4^2 = 0 \tag{3.22}$$

Physical Behavior of the Model. Equation (3.21) can be written as

$$\beta = \frac{m_5}{4}\left[\sinh\left(\frac{\eta + m_2}{2}\right)\right]^{m_3 - 2}\left[\cosh\left(\frac{\eta + m_2}{2}\right)\right]^{-(m_3 + 2)} \tag{3.23}$$

which suggests that when $\eta \to -m_2$:

(i) $\beta \to 0$ if $m_3 > 2$.
(ii) $\beta \to m_5/4$ if $m_3 = 2$.
(iii) $\beta \to \infty$ if $m_3 < 2$.

The Ricci scalar is

$$R=-\frac{3}{32}m_5^2\left[\sinh\left(\frac{\eta+m_2}{2}\right)\right]^{2(m_3-2)}\left[\cosh\left(\frac{\eta+m_2}{2}\right)\right]^{-2(m_3+2)} \tag{3.24}$$

We define the expansion scalar Θ and shear scalar σ (Raychaudhuri, 1955) as

$$\Theta=\frac{3\dot{V}}{V}, \qquad \text{where} \quad V^3=(-g_{11}g_{22}g_{33})^{1/2}$$

$$\Theta=\frac{1}{R_2}\left(\frac{R_1'}{R_1}+\frac{2R_2'}{R_2}\right)$$

$$=\frac{m_4}{4m_1}[2\cosh(\eta+m_2)-m_3]\left[\operatorname{sech}\left(\frac{\eta+m_2}{2}\right)\right]^4 \times\left[\tanh\left(\frac{\eta+m_2}{2}\right)\right]^{m_3-2} \tag{3.25}$$

$$\sigma^2=\frac{1}{12}\left[\left(\frac{\dot{g}_{11}}{g_{11}}-\frac{\dot{g}_{22}}{g_{22}}\right)^2+\left(\frac{\dot{g}_{22}}{g_{22}}-\frac{\dot{g}_{33}}{g_{33}}\right)^2+\left(\frac{\dot{g}_{33}}{g_{33}}-\frac{\dot{g}_{11}}{g_{11}}\right)^2\right]$$

$$\sigma^2=\frac{m_4^2}{6m_1^2}\left[\operatorname{sech}\left(\frac{\eta+m_2}{2}\right)\right]^8\left[\tanh\left(\frac{\eta+m_2}{2}\right)\right]^{2m_3-4} \times[2m_3-\cosh(\eta+m)]^2 \tag{3.26}$$

The cosmological parameters H (Hubble's parameter) and q (deceleration parameter) are

$$H=\frac{\dot{V}}{V}=\frac{1}{3R_2}\left(\frac{R_1'}{R_1}+\frac{2R_2'}{R_2}\right)$$

$$H=\frac{m_4}{12m_1}[2\cosh(\eta+m_2)-m_3]\left[\operatorname{sech}\left(\frac{\eta+m_2}{2}\right)\right]^4 \times\left[\tanh\left(\frac{\eta+m_2}{2}\right)\right]^{m_3-2} \tag{3.27}$$

$$q=-\frac{V\ddot{V}}{\dot{V}^2}$$

$$q=2-6[\sinh(\eta+m_2)]^2[2\cosh(\eta+m_2)-m_3]^{-2} \tag{3.28}$$

Case II. Matter-Filled Universe

For a matter-filled universe we have not been able to find solutions in the case of dust- and radiation-dominated universe. However, the solution for a superdense universe (i.e., $p=\rho$) can be obtained.

Taking $p=\rho$, viz. $\lambda=2$, we have found that the solutions are the same as those for an empty universe, but equation (3.11) gives

$$p=\rho=-\frac{3}{2\chi}\beta^2+\frac{m}{\chi}[\operatorname{cosech}(\eta+m_2)]^4\left[\tan\left(\frac{\eta+m_2}{2}\right)\right]^{2m_3} \tag{3.29}$$

where m is another arbitrary constant related to the others through

$$mm_1^2+(m_3^2-1)m_4^2=0 \tag{3.30}$$

Equation (3.29) suggests that pressure and density depend on the time function η as well as the gauge function β.

Physical Behavior of the Model. In the case of a Zeldovich fluid the Ricci scalar becomes

$$R=-\frac{m}{8}\left[\sinh\left(\frac{\eta+m_2}{2}\right)\right]^{2(m_3-2)}\left[\cosh\left(\frac{\eta+m_2}{2}\right)\right]^{-2(m_3+2)} \tag{3.31}$$

It is clear that when $\eta\to-m_2$:

(i) $R\to 0$ if $m_3>2$.
(ii) $R\to -m/8$, if $m_3=2$.
(iii) $R\to-\infty$ if $m_3<2$.

The relative anisotropy is given by

$$\frac{\sigma^2}{\rho}=\frac{8m_4^2}{3m_1^2}\frac{2m_3-\cosh(\eta+m_2)}{m-24\beta^2[\sinh\frac{1}{2}(\eta+m_2)]^{2(2-m_3)}[\cosh\frac{1}{2}(\eta+m_2)]^{2(2+m_3)}} \tag{3.32}$$

When $\eta\to-m_2$:

(i) $\sigma^2/\rho\to 0$ if $m_3>2$.
(ii) $\sigma^2\propto\rho$ if $m_3\leq 2$.

3.2. $\beta=$ const

Case I. Empty Universe

Considering the displacement vector β as a constant in the case of an empty universe, we find that equations (3.5) and (3.6) reduce to

$$\frac{\dot R_1}{R_1}+\frac{2\dot R_2}{R_2}=0 \tag{3.33}$$

$$\frac{\ddot R_1}{R_1}+\frac{2\dot R_1\dot R_2}{R_1R_2}=0 \tag{3.34}$$

Using equation (3.33), from equation (3.34) we obtain the solution as

$$R_1=\exp(at+b) \tag{3.35}$$

where a and b are arbitrary constants.

From equations (3.33) and (3.35) we have

$$R_2 = C \exp\left(-\frac{at+b}{2}\right) \tag{3.36}$$

With the help of equations (3.35) and (3.36), equation (3.3) gives a relation between constants a and β,

$$a^2 = -\beta^2 \tag{3.37}$$

For a to be real, β must be imaginary.

The solution in metric form can be written as

$$ds^2 = dt^2 - \exp[2(at+b)]\, dx^2 - \exp[-(at+b)](d\theta^2 + \sinh^2\theta\, d\phi^2) \tag{3.38}$$

Physical Behavior of the Model. The Ricci scalar R is

$$R = 3a^2 - 2\exp(at+b) \tag{3.39}$$

When (i) $t \to -b/a$, $R \to 3a^2 - 2$, and (ii) $t \to \infty$, $R \to -\infty$.

The expansion and shear scalars are

$$\Theta = 0 \tag{3.40}$$

$$\sigma^2 = \tfrac{3}{2}a^2 \tag{3.41}$$

Hence in this model there is no expansion.

Case II. Matter-Filled Universe

In this case also we have only found the solution for a Zeldovich fluid. The solutions are the same as in case II of the matter-filled universe in Section 3.1.

4. KANTOWSKI–SACHS SPACE-TIME

4.1. $\beta = \beta(t)$

The Kantowski–Sachs metric takes the form

$$ds^2 = dt^2 - R_1^2\, dr^2 - R_2^2(d\theta^2 + \sin^2\theta\, d\phi) \tag{4.1}$$

which can be obtained from the general form of the metric (2.5) when $f(\theta) = \sin\theta$.

The corresponding field equations (2.6)-(2.9) for the metric (4.1) reduce to

$$\frac{2\ddot{R}_2}{R_2}+\left(\frac{\dot{R}_2}{R_2}\right)^2+\frac{1}{R_2^2}=-\chi p-\frac{3}{4}\beta^2 \tag{4.2}$$

$$\frac{\ddot{R}_1}{R_1}+\frac{\ddot{R}_2}{R_2}+\frac{\dot{R}_1\dot{R}_2}{R_1R_2}=-\chi p-\frac{3}{4}\beta^2 \tag{4.3}$$

$$\left(\frac{2\dot{R}_1}{R_1}+\frac{\dot{R}_2}{R_2}\right)\frac{\dot{R}_2}{R_2}+\frac{1}{R^2}=\chi\rho+\frac{3}{4}\beta^2 \tag{4.4}$$

$$\chi\dot{\rho}+\frac{3}{2}\beta\dot{\beta}+\left[\chi(p+\rho)+\frac{3}{2}\beta^2\right]\left(\frac{\dot{R}_1}{R_1}+\frac{2\dot{R}_2}{R_2}\right)=0 \tag{4.5}$$

Adding equation (4.4) to twice equation (4.3), then subtracting equation (4.2), we obtain

$$\frac{\ddot{R}_1}{R_1}+\frac{2\dot{R}_1\dot{R}_2}{R_1R_2}=\frac{\chi}{2}(\rho-p) \tag{4.6}$$

From equations (4.3) and (4.4), we get

$$\frac{\ddot{R}_1}{R_1}+\frac{\ddot{R}_2}{R_2}+\left(\frac{3\dot{R}_1}{R_1}+\frac{\dot{R}_2}{R_2}\right)\frac{\dot{R}_2}{R_2}+\frac{1}{R_2^2}=\chi(\rho-p) \tag{4.7}$$

Using the transformations (3.8) and (3.9) and equation of state (2.4), we can write equations (4.4)-(4.7) as

$$\left(\frac{2R_1'}{R_1}+\frac{R_2'}{R_2}\right)\frac{R_2'}{R_2}+1=\left(\chi\rho+\frac{3}{2}\beta^2\right)R_2^2 \tag{4.8}$$

$$\chi\rho'+\frac{3}{2}\beta\beta'+\left[\chi(p+\rho)+\frac{3}{2}\beta^2\right]\left(\frac{h'}{h}+\frac{R_2'}{R_2}\right)=0 \tag{4.9}$$

$$\frac{R_1''}{R_1}-\left(\frac{R_1'}{R_1}\right)^2+\frac{h'R_1'}{hR_1}=\frac{\chi}{2}(2-\lambda)\rho \tag{4.10}$$

$$\frac{h''}{h}+1=\chi(2-\lambda)\rho \tag{4.11}$$

In the present form the field equations are difficult to solve. We therefore consider some cases of physical interest.

Case I. Empty Universe $(p = \rho = 0)$

In this case the set of field equations (4.8)-(4.11) reduce to

$$\left(\frac{2R_1'}{R_1}+\frac{R_2'}{R_2}\right)\frac{R_2'}{R_2}+1=\frac{3}{2}\beta^2R_2^2 \tag{4.12}$$

$$\frac{\beta'}{\beta}+\frac{h'}{h}+\frac{R_2'}{R_2}=0 \tag{4.13}$$

$$\frac{R_1''}{R_1}-\left(\frac{R_1'}{R_1}\right)^2+\frac{h'R'}{hR_1}=0 \tag{4.14}$$

$$h''+h=0 \tag{4.15}$$

Equation (4.15) yields the solution

$$h=C_1\sin(\eta+C_2) \tag{4.16}$$

Using equation (4.16), we get from (4.14)

$$R_1=C_4\left[\tan\left(\frac{\eta+C_2}{2}\right)\right]^{C_3} \tag{4.17}$$

According to assumption

$$R_2=\frac{h}{R_1}$$

Hence

$$R_2=\frac{C_1}{C_4}\sin(\eta+C_2)\left[\cot\left(\frac{\eta+C_2}{2}\right)\right]^{C_3} \tag{4.18}$$

With the help of equations (4.16) and (4.18) we get from equation (4.13)

$$\beta=C_5[\operatorname{cosec}(\eta+C_2)]^2\left[\tan\left(\frac{\eta+C_2}{2}\right)\right]^{C_3} \tag{4.19}$$

Hence C_i, $i=1,2,\ldots,5$, are arbitrary constants. Equation (4.12) is satisfied only when

$$3C_1^2C_5^2+4(C_3^2-1)C_4^2=0 \tag{4.20}$$

Physical Behavior of the Model. Equation (4.19) can be written as

$$\beta=\frac{C_5}{4}\left[\sin\left(\frac{\eta+C_2}{2}\right)\right]^{C_3-2}\left[\cos\left(\frac{\eta+C_2}{2}\right)\right]^{-(C_3+2)} \tag{4.21}$$

which shows that when $\eta\to -C_2$:

(i) $\beta\to 0$ if $C_3>2$.

(ii) $\beta \to C_5/4$ if $C_3 = 2$.
(iii) $\beta \to \infty$ if $C_3 < 2$.

The Ricci scalar is

$$R = -\frac{3C_5^2}{32}\left[\sin\left(\frac{\eta + C_2}{2}\right)\right]^{2(C_3-2)}\left[\cos\left(\frac{\eta + C_2}{2}\right)\right]^{-2(C_3+2)} \tag{4.22}$$

When $\eta \to -C_2$:

(i) $R \to 0$ if $C_3 > 2$.
(ii) $R \to -3C_5^2/32$ if $C_3 = 2$.
(iii) $R \to -\infty$ if $C_3 < 2$.

The scalars of expansion and shear are given by

$$\Theta = \frac{C_4}{4C_1}[2\cos(\eta + C_2) - C_3]\left[\sec\left(\frac{\eta + C_2}{2}\right)\right]^4 \times \left[\tan\left(\frac{\eta + C_2}{2}\right)\right]^{C_3-2} \tag{4.23}$$

$$\sigma^2 = \frac{C_4^2}{6C_1^2}[2C_3 - \cos(\eta + C_2)]^2\left[\sec\left(\frac{\eta + C_2}{2}\right)\right]^8 \times \left[\tan\left(\frac{\eta + C_2}{2}\right)\right]^{2C_3-4} \tag{4.24}$$

The Hubble parameter H and deceleration parameter q are

$$H = \frac{C_4}{12C_1}[2\cos(\eta + C_2) - C_3]\left[\sec\left(\frac{\eta + C_2}{2}\right)\right]^4 \times \left[\tan\left(\frac{\eta + C_2}{2}\right)\right]^{C_3-2} \tag{4.25}$$

$$q = 2 + 6[\sin(\eta + C_2)]^2[2\cos(\eta + C_2) - C_3]^2 \tag{4.26}$$

Case II. Matter-Filled Universe

In this case we have not been able to find solutions for a dust- and radiation-dominated universe. However, for a superdense universe the solution can be obtained.

Considering $p = \rho$, viz. $\lambda = 2$, we have found that the solutions are the same as the solutions of case I, but equation (4.9) leads to

$$p = -\frac{3}{4\chi}\beta^2 + \frac{CC_4^2}{\chi C_1^4[\sin(\eta + C_2)]^4[\cot\frac{1}{2}(\eta + C_2)]^{2C_3}} \tag{4.27}$$

Here C is an arbitrary constant related to other constants through

$$C+(C_3^2-1)C_1^2C_4=0 \tag{4.28}$$

From equation (4.27) it is clear that pressure and density both depend on the time function η as well as the gauge function β.

Physical Behavior of the Model. The Ricci scalar is

$$R=-\frac{CC_4}{C_1^4}\left[\sin\left(\frac{\eta+C_2}{2}\right)\right]^{2(C_3-2)}\left[\cos\left(\frac{\eta+C_2}{2}\right)\right]^{-2(C_3+2)} \tag{4.29}$$

It is clear that when $\eta\to -C_2$:

(i) $R\to 0$ if $C_3>2$.
(ii) $R\to -CC_4/C_1^4$ if $C_3=2$.
(iii) $R\to\infty$ if $C_3<2$.

The relative anisotropy is given by

$$\frac{\sigma^2}{\rho}=\frac{8}{3}\,\frac{\chi C_1^2C_4^2[2C_3-\cos(\eta+C_2)]}{CC_4-24C_1^4\beta^2[\sin\frac{1}{2}(\eta+C_2)]^{2(2-C_3)}[\cos\frac{1}{2}(\eta+C_2)]^{2(2+C_3)}} \tag{4.30}$$

When $\eta\to -C_2$:

(i) $\sigma^2/\rho\to 0$ if $C_3>2$.
(ii) $\sigma^2\propto\rho$ if $C_3\le 2$.

4.2. $\beta=$ const

Empty Universe

Taking β as a constant in an empty universe ($p=\rho=0$), the set of field equations (4.4)-(4.7) reduce to

$$\left(\frac{2\dot R_1}{R_1}+\frac{\dot R_2}{R_2}\right)\frac{\dot R_2}{R_2}+\frac{1}{R_2^2}=\frac{3}{4}\beta^2 \tag{4.31}$$

$$\frac{\dot R_1}{R_2}+\frac{2\dot R_2}{R_2}=0 \tag{4.32}$$

$$\frac{\ddot R_1}{R_1}+\frac{2\dot R_1\dot R_2}{R_1R_2}=0 \tag{4.33}$$

$$\frac{\ddot R_1}{R_1}+\frac{\ddot R_2}{R_2}+\frac{\dot R_2}{R_2}\left(\frac{3\dot R_1}{R_1}+\frac{\dot R_2}{R_2}\right)+\frac{1}{R_2^2}=0 \tag{4.34}$$

From equations (4.31) and (4.34), we get

$$\frac{\ddot R_1}{R_1}+\frac{\ddot R_2}{R_2}+\frac{\dot R_1\dot R_2}{R_1R_2}=-\frac{3}{4}\beta^2 \tag{4.35}$$

Using equation (4.32) in equation (4.33) and then integrating, we obtain

$$R_1 = \exp(a_1 t + b_1) \tag{4.36}$$

Here a_1 and b_1 are arbitrary constants.

With the help of (4.36), from (4.32) we get

$$R_2 = a_2 \exp\left(-\frac{a_1 t + b_1}{2}\right) \tag{4.37}$$

where a_2 is another arbitrary constant.

Equation (4.35) is satisfied only when

$$a_1^2 = -\beta^2 \tag{4.38}$$

For a_1 to be real, β must be imaginary.

In this model the expansion scalar vanishes and the shear scalar becomes a constant. The particle horizon exists.

Case II. Matter-Filled Universe

In this case the solutions are the same as in case II of Section 4.1.

APPENDIX. A BRIEF NOTE ON LYRA'S GEOMETRY

Lyra (1951) proposed a new modification of Riemannian geometry which removes the nonintegrability condition of the length of the vector under parallel transport.

Lyra defined the displacement vector PP' between two neighboring points $P(x^\mu)$ and $P'(x^\mu + dx^\mu)$ by its components $x^0\, dx^\mu$, where $x^0 = x^0(x^\mu)$ is a gauge function. The coordinate system x^μ and the gauge function x^0 together form a reference system (x^0, x^μ). The transformation to a new reference system $(\bar{x}^0, \bar{x}^\mu)$ is given by

$$\bar{x}^0 = \bar{x}^0(x^0, x^\mu), \qquad \bar{x}^\mu = \bar{x}^\mu(x^\mu) \tag{A.1}$$

with

$$\frac{\partial \bar{x}^0}{\partial x^0} \neq 0 \qquad \text{and Jacobian} \qquad \left|\frac{\partial \bar{x}^\mu}{\partial x^\mu}\right| \neq 0$$

The connections $\Gamma^\alpha_{\mu\nu}$ are given by

$$\Gamma^\alpha_{\mu\nu} = (x^0)^{-1}\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} + \frac{1}{2}(\delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\mu\nu}\phi^\alpha) \tag{A.2}$$

where the $\{^{\alpha}_{\mu\nu}\}$ are defined in terms of the metric tensor $g_{\mu\nu}$ as in Riemannian geometry and ϕ_μ is a displacement vector field.

It is shown by Lyra (1951) and Sen (1957) that in any general reference system the vector field quantities ϕ appear as a natural consequence of the introduction of the gauge function x^0 into the structureless manifold.

The metric in Lyra's geometry is given by

$$ds^2 = g_{\mu\nu} x^0\, dx^\mu x^0\, dx^\nu \tag{A.3}$$

and is invariant under both coordinate and gauge transformations.

The infinitesimal parallel transfer of a vector is given by

$$d\xi^\alpha = -\tilde{\Gamma}^\alpha_{\mu\nu} \xi^\mu x^0\, dx^\nu \tag{A.4}$$

where

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \tfrac{1}{2}\delta^\alpha_\mu \phi_\nu$$

The $\tilde{\Gamma}^\alpha_{\mu\nu}$ are not symmetric, but the $\Gamma^\alpha_{\mu\nu}$ are symmetric in μ and ν. The length of a vector does not change under parallel transport, unlike in Weyl's geometry. The curvature tensor ${}^*R^\alpha{}_{\lambda\mu\nu}$ is defined by

$$\begin{aligned} {}^*R^\alpha{}_{\lambda\mu\nu} = \frac{1}{(x^0)^2} \bigg[& \frac{\partial}{\partial x^\mu}(x^0 \tilde{\Gamma}^\alpha_{\lambda\nu}) - \frac{\partial}{\partial x^\nu}(x^0 \tilde{\Gamma}^\alpha_{\lambda\mu}) \\ & + x^0 \tilde{\Gamma}^\alpha_{\beta\mu} x^0 \tilde{\Gamma}^\beta_{\lambda\nu} - x^0 \tilde{\Gamma}^\alpha_{\beta\nu} x^0 \tilde{\Gamma}^\beta_{\lambda\mu} \bigg] \end{aligned} \tag{A.5}$$

The curvature scalar, obtained by contraction of equation (A.5), is

$$\begin{aligned} {}^*R = {} & (x^0)^{-2} R + 3(x^0)^{-1} \phi^\mu_{;\mu} + \tfrac{3}{2}\phi^\mu \phi_\mu \\ & + 2(x^0)^{-1} \frac{\partial}{\partial x^\mu}[\log(x^0)^2]\, \phi^\mu \end{aligned} \tag{A.6}$$

The volume integral is given by

$$I = \int L(-g)^{1/2} (x^0)^4\, d^4x \tag{A.7}$$

where d^4x is the volume element and L is a scalar invariant.

If we use a normal gauge, i.e., $x^0 = 1$ (Sen, 1957) and following Halford (1970) we let $L = {}^*R$, then equations (A.6) and (A.7) become, respectively,

$${}^*R = R + 3\phi^\mu{}_{;\mu} + \tfrac{3}{2}\phi^\mu \phi_\mu \tag{A.8}$$

$$I = \int {}^*R(-g)^{1/2}\, d^4x \tag{A.9}$$

The field equations are obtained from the variational principle

$$\delta(I + J) = 0 \tag{A.10}$$

where I is as given by equation (A.9) and J is related to the Lagrangian density $\mathscr{L}$ of matter by

$$J = \int \mathscr{L}(-g)\, d^4x \tag{A.11}$$

The field equations are thus (using $\chi = 8\pi G/c^2$)

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} + \tfrac{3}{2}\phi_\mu\phi_\nu - \tfrac{3}{4}g_{\mu\nu}\phi_\alpha\phi^\alpha = -\chi T \tag{A.12}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. For further details we refer to Bhamra (1974), Halford (1970), Sen (1957, 1960), and Sen and Dunn (1971).

ACKNOWLEDGMENTS

The authors would like to place on record their sincere thanks to the CSIR, New Delhi, for financial support to this work through Research Project No. 25(31)/87-EMR-II (T.S.) and CSIR(SRF) vide No. 9/13(348)90EMR-I (G.P.S.).

REFERENCES

Beesham, A. (1986). *Astrophysics and Space Science*, **127**, 189.
Bhamra, K. S. (1974). *Australian Journal of Physics*, **27**, 541.
Cohen, M., and Defrise, L. (1968). *Communications in Mathematical Physics*, **11**, 56.
Grøn, Ø. (1986). *Journal of Mathematical Physics*, **27**, 1490.
Halford, W. D. (1970). *Australian Journal of Physics*, **23**, 863.
Lorenz, D. (1982). *Journal of Physics A: Mathematical and General*, **15**, 2997.
Lorenz, D. (1983). *Journal of Physics A: Mathematical and General*, **16**, 575.
Lyra, G. (1951). *Mathematische Zeitschrift*, **54**, 52.
Matravers, D. R. (1988). *General Relativity and Gravitation*, **20**, 279.
Moussiaux, A., Tombal, P., and Demaret, J. (1981). *Journal of Physics A: Mathematical and General*, **14**, L277.
Raychaudhuri, A. K. (1955). *Physical Review*, **98**, 1123.
Sen, D. K. (1957). *Zeitschrift für Physik*, **149**, 311.
Sen, D. K. (1960). *Canadian Mathematical Bulletin*, **3**, 255.
Sen, D. K., and Dunn, K. A. (1971). *Journal of Mathematical Physics*, **12**, 578.
Singh, T., and Singh, G. P. (1991*a*). *Journal of Mathematical Physics*, **32**, 2456.
Singh, T., and Singh, G. P. (1991*b*). *Astrophysics and Space Science*, **182**, 189.
Soleng, H. H. (1987). *General Relativity and Gravitation*, **19**, 1213.
Stewart, J. M., and Ellis, G. F. R. (1968). *Communications in Mathematical Physics*, **11**, 56.
Vajk, J. P., and Eltgroth, P. G. (1970). *Journal of Mathematical Physics*, **11**, 2212.
Weber, E. (1984). *Journal of Mathematical Physics*, **25**, 3279.
Weber, E. (1985). *Journal of Mathematical Physics*, **26**, 1308.
Weyl, H. (1918). *Sitzungsberichte der Preussichen Academie der Wissenschaften*, **1918**, 465.